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LETTER TO THE EDITOR

Zeros of the partition function for the two-dimensional Ising model

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Abstract. The zeros of the partition function for the zero field two-dimensional Ising model in the complex temperature plane are represented in terms of the eigenvalues of the transfer matrix. The boundaries of the limiting distribution have a particular simple representation in terms of the correlation range along the two principal axes of the quadratic lattice. Examples of the distribution in the complex plane of $z=\exp(-2K)$ are given for the anisotropic model when the horizontal and vertical interactions are in integer ratio.

In two recent papers Saarloos and Kurtze (1984) and Stephenson and Couzens (1984) have considered the limiting distribution of the zeros of the partition function of the Ising model in zero magnetic field on the two-dimensional quadratic lattice. A number of authors have also investigated such limiting distributions for the scalar Potts models on hierarchical lattices; these are the Julia sets of rational complex maps (Derrida *et al* 1983, Itzykson and Luck 1983, Derrida *et al* 1985). The purpose of this letter is to introduce a representation of the limiting distribution in the Ising model case which readily enables a general view of the distribution to emerge as simply connected regions in the complex temperature plane. In this new representation the location of all the zeros is related to the spectral properties of the transfer matrix in the thermodynamic limit (Onsager 1944, Kaufman 1949).

If the reduced interactions K and K' represent interactions along the horizontal and vertical bonds respectively of the quadratic lattice the exact partition function of an $m\times n$ lattice can be expressed in terms of four continued products each of the form

$$\prod_{r=1}^{n}\prod_{t=1}^{m}\{cc'-s'\cos\phi'_r-s\cos\phi_t\} \tag{1}$$

where c and s are $\cosh 2K$ and $\sinh 2K$ respectively and the angles ϕ'_r and ϕ_t are chosen from the sets $r\pi/n$, $r=1,2,\ldots,2n$ and $t\pi/m$, $t=1,2,\ldots,2m$ respectively. Only one of these products is needed to describe the asymptotic distribution obtained in the limit $m, n\to\infty$ (Brascamp and Kunz 1974, Katsura 1967, Abe and Katsura 1970); and in the isotropic case $K'=K$ it is trivial to show that the zeros are dense on the unit circle

$$s=e^{i\psi}. \tag{2}$$

When $K\neq K'$ the limiting distribution has a more complex structure, the zeros are distributed densely in a finite area of the complex plane and are generated by the roots of

$$cc'-s'\cos\phi'-s\cos\phi=0, \qquad 0\leqslant\phi',\ \phi\leqslant 2\pi. \tag{3}$$

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Saarloos and Kurtze, and Stephenson and Couzens considered the cases $K'=\alpha K$ with an illustration of the location of the zeros in the complex s plane with $\alpha=3$. The distribution depends sensitively upon α, Saarloos and Kurtze simply expanded the factors (3) in terms of a polynomial in s and explored the roots of (3) in the form $s(\phi, \phi')$ for $\alpha=3$. Stephenson and Couzens chose the variable

$$w=\sinh\tfrac{1}{2}(K+K')/\cosh\tfrac{1}{2}(K-K')=S_K/C_k \tag{4}$$

by writing (3) in the form

$$S_K C_k(w+w^{-1}-(\cos\phi+\cos\phi')-(C_K S_k/S_K C_k)(\cos\phi-\cos\phi'))=0 \tag{5}$$

and again explored the roots $s(\phi, \phi')$ for $\alpha=3$.

An alternative representation of (3) is readily obtained in terms of the eigenvalues of the transfer matrix (Onsager 1944, Kaufman 1949, Domb 1960) which for an $n\times\infty$ strip of the square lattice are generated by the two sets

$$\lambda_i^+=(2\sinh 2K)^{n/2}\exp\tfrac{1}{2}\{\pm\gamma_1\pm\gamma_3\pm\ldots\pm\gamma_{2n-1}\} \tag{6}$$

$$\lambda_i^-=(2\sinh 2K)^{n/2}\exp\tfrac{1}{2}\{\pm\gamma_2\pm\gamma_4\pm\ldots\pm\gamma_{2n-2}\pm\gamma_0\} \tag{7}$$

where any combination of the γ_r's with an even number of negative signs corresponds to one of the 2^n eigenvalues and

$$\cosh\gamma_r=cc'/s-s'\cos\phi'_r/s, \qquad \phi'_r=r\pi/n \qquad r=1,2,\ldots,2n. \tag{8}$$

Hence all the zeros generated by the factors in (1) can be expressed in the form

$$e^{\gamma_r}=e^{i\phi_i} \tag{9}$$

and in the continuum of the thermodynamic limit we can write this in the form

$$e^{\gamma_{\phi'}}=e^{i\phi}, \qquad 0\leqslant\phi, \phi'\leqslant 2\pi \tag{10}$$

hence *all* the zeros of the partition function can be viewed as mappings of these unit circles into a suitable complex temperature variable. The natural boundaries of the distribution are in fact determined by the correlation lengths along the two principal axes. Two such boundaries are the curves $s(\phi, 0)$ and $s(0, \phi')$ in the s plane, which are mappings of the two circles

$$\exp(\gamma_0(K, K'))=e^{i\phi} \tag{11a}$$

and

$$\exp(\gamma_0(K', K))=e^{i\phi}, \qquad 0\leqslant\phi\leqslant 2\pi \tag{11b}$$

where for this model $\gamma_0(K, K')=-\kappa(K, K')$, the inverse correlation length in the horizontal direction. Thus in terms of the correlation length these two boundaries of the distribution are given by the equations

$$cc'-s'-s\cosh\kappa_h(K, K')=0, \qquad \kappa_h=i\phi \tag{12a}$$

and

$$cc'-s-s'\cosh\kappa_v(K, K')=0, \qquad \kappa_v=i\phi, \qquad 0\leqslant\phi\leqslant 2\pi \tag{12b}$$

where $\kappa_v=\kappa_h(K', K)$ is the inverse correlation length in the vertical direction. The two remaining boundaries $s(\phi, \pi)$ and $s(\pi, \phi')$ are simply obtained under the antiferromagnetic transformation $K'\to -K'$. It is very tempting to seek a relationship between the correlation length in any arbitrary lattice direction and subsets of zeros within the limiting distribution, indeed if $K = K'$ and we look at the subset of zeros given by $\phi' = \phi$ then (3) takes the form

$$c^2 - 2s \cosh \kappa_{\pi/4}(K) = 0, \qquad \kappa_{\pi/4} = \mathrm{i}\phi \qquad 0 \leq \phi \leq 2\pi \tag{13}$$

where $\kappa_{\pi/4}$ is the inverse correlation length in the diagonal direction (see Domb 1960). However (12) and (13) are all that seems possible along these lines. A 'tentative' attempt to obtain the correlation range in an arbitrary direction was made by Onsager (1944) in his original paper, but as was pointed out by Fisher and Burford (1967) Onsager's generalisation of (12) and (13) is incorrect. In summary the main observation on the distribution of zeros is that they are all generated by the class of circles in (10), which clearly can also be expressed in terms of ratios of the eigenvalues (6) and (7) in the thermodynamic limit.

In order to obtain an actual view of the limiting distribution we can first examine its boundaries; the two boundaries (11*a*) and (11*b*) yield the two equations

$$z^\alpha(1+z)/(1-z) = \mathrm{e}^{\mathrm{i}\phi}, \qquad K' = \alpha K \tag{14a}$$

and

$$z(1+z^\alpha)/(1-z^\alpha) = \mathrm{e}^{\mathrm{i}\phi}, \qquad 0 \leq \phi \leq 2\pi \tag{14b}$$

where z is the complex variable e^{-2K} and α is real. The roots of (14*a*) lie on the curve whose polar equation is

$$\cos\theta = -\frac{(r^{2\alpha}-1)}{(r^{2\alpha}+1)}\frac{(1+r^2)}{2r} \tag{15}$$

and the transformation $K' \to -K'$ in effect reflects this curve in the imaginary z axis. For $\alpha = 1$ these two curves are the two circles

$$z = \pm 1 + \sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta} \tag{16}$$

obtained by Fisher (1965). The curves generated by (14*a*) up to $\alpha = 10$ are shown in figure 1, the limiting case $\alpha \to \infty$ is of course the unit circle which is the distribution of the one-dimensional Ising model.

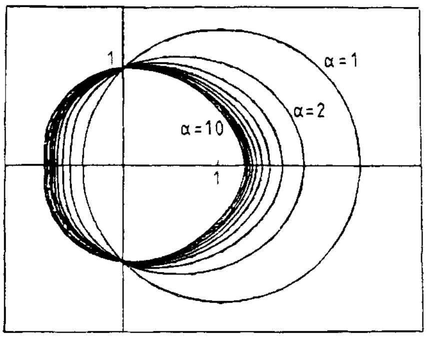

Figure 1. Solutions to equation (14*a*) for $\alpha = 1$ to 10. For a given α the curve here together with its reflection in the imaginary z axis are two of the boundaries of the limiting distribution.

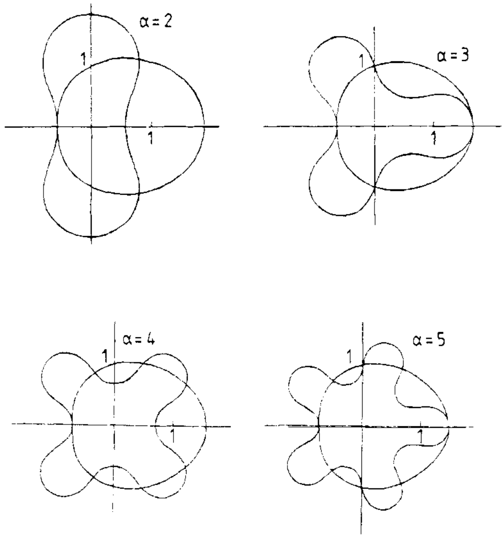

Figure 2. The two boundaries given by equations (14a) and (18) for $\alpha = 2$, 3, 4, and 5.

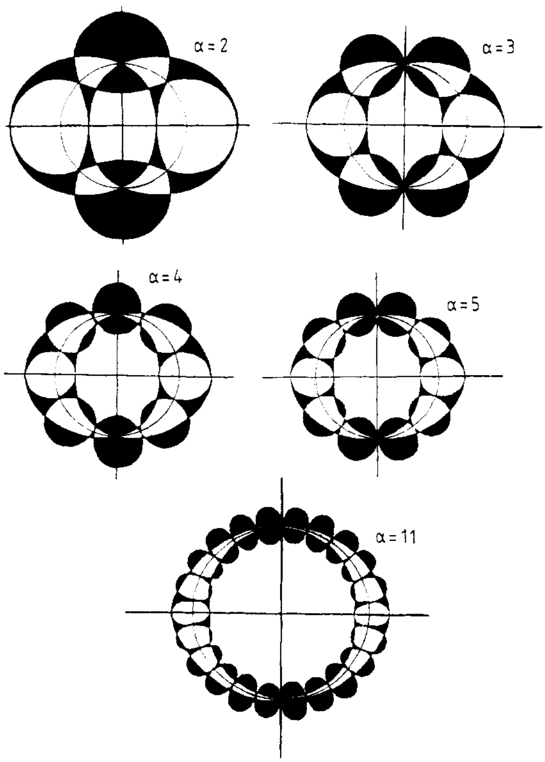

Figure 3. The limiting distribution of zeros (shaded regions) for $\alpha = 2$, 3, 4, 5 and 11. The four boundaries are (i) solutions to (14a) (figure 1) and their reflections in the imaginary z axis and (ii) solutions to (18) (shown in figure 2) and their rotation through the angle π/α. The unit circle is also shown.

The second boundary (14*b*) has a more complicated structure; in terms of the variable

$$q = z^{\alpha} = r_q\, e^{i\theta}, \tag{17}$$

the polar equation for q is given by (15) under the transformation $\alpha \to 1/\alpha$ and

$$z = r_q^{1/\alpha}\, e^{i\theta/\alpha}. \tag{18}$$

The final boundary obtained under the transformation of (18) under $K' \to -K'$ is simply a rotation of the curve (18) through the angle π/α. The curves (18) for $\alpha = 2$, 3, 4 and 5 are shown in figure 2 where they are superimposed on the corresponding curves of (15). For α an integer the two boundaries generated by (18) and its corresponding rotation will intersect at 2α points on the unit circle in the z plane with the angle between the radial vectors at neighbouring intersection points being π/α. This will generate cusp-like contacts on the unit circle; in fact the intersections of all the four boundaries will produce a distribution of zeros in the z plane in the form of a necklace of bounded regions simply connected through cusp-like contacts.

Only a finite number of zeros can occur on the unit circle (Saarloos and Kurtze 1984) and this can serve to locate those regions in which zeros cannot be dense, alternatively starting at the points on the real axis and moving into the complex plane counting an even number of crossings with the four boundary curves readily locates those areas where the zeros lie. The distributions for $\alpha = 2$, 3, 4, 5 and 11 are shown in figure 3. The effect of increasing the anisotropy is to increase the number of zeros on the unit circle and in the limit the whole distribution will collapse onto this circle recovering the one-dimensional distribution.

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